Aerodynamic Stability of a Coasting Vehicle Rapidly Ascending through the Atmosphere

GEORGE W. STONE*
Sandia Corporation, Albuquerque, N. Mex.

This paper presents an analysis of the aerodynamic stability of a coasting vehicle exiting the atmosphere at high speeds. With the assumption of linear theory and constant aerodynamic coefficients, an approximate solution and an exact solution to the equations of motion are obtained. Attention is focused upon the approximate solution since the exact solution involves confluent hypergeometric functions with complex coefficients that are not readily tractable. The approximate solution is compared with six-degree-of-freedom computer results.

Nomenclature

a = parameter in confluent hypergeometric function
A = reference area, ft²
B_0 = constant of integration
B_1 = constant of integration
c = parameter in confluent hypergeometric function
C_D = drag coefficient
$C_{M\alpha}$ = static moment coefficient slope, 1/rad
$C_{M_q} + C_{M\dot{\alpha}}$ = damping moment coefficient, 1/rad
$C_{Mp\alpha}$ = Magnus moment coefficient, 1/rad²
$C_{N\alpha}$ = normal force coefficient slope, 1/rad
D = damping parameter = $(A/2m)[C_{N\alpha} - 2C_D - k_t^{-2}(C_{M_q} + C_{M\dot{\alpha}})]$
E = envelope of motion, rad or deg
G = gyroscopic parameter = $(p/V)(I_x/I_y)$
h = altitude, ft
i = $(-1)^{1/2}$
I = moment of inertia, slug-ft²
I_x = roll moment of inertia, slug-ft²
I_y = transverse moment of inertia, slug-ft²
I_z = transverse moment of inertia, slug-ft²
k = density gradient parameter = $\beta \sin\gamma$
k_a = nondimensional axial radius of gyration = $(I_x/ml^2)^{1/2}$
k_t = nondimensional transverse radius of gyration = $(I/ml^2)^{1/2} = (I_y/ml^2)^{1/2} = (I_z/ml^2)^{1/2}$
K = constant of integration
l = reference length, ft
m = mass, slugs
M = static moment parameter = $AlC_{M\alpha}/2I$
p = roll rate, rad/sec
S = distance traveled, ft
T = Magnus moment parameter = $(A/2m)(C_{N\alpha} - C_D + k_a^{-2}C_{Mp\alpha})$
V = velocity, fps
$\tilde{\alpha}$ = angle of attack, rad
β = exponent in density-altitude function
$\tilde{\beta}$ = angle of yaw, rad
γ = flight path angle, deg
δ = error of *WKB* method
θ = phase-shift function
$\tilde{\xi}$ = complex angle of yaw
ρ = density, slugs/ft³
ϕ = phase angle or argument
Φ = confluent hypergeometric function

Received September 23, 1965; revision received May 27, 1966. This work was supported by the U. S. Atomic Energy Commission.

* Staff Member, Aero-Thermodynamics Department, Sandia Laboratory. Member AIAA

Introduction

THE problem of vehicle stability has long plagued the aerodynamicist and, as the performance of aerodynamic vehicles has increased, many new stability problems have received attention. One such problem is the effect of a gradient in dynamic pressure on the oscillatory motion of a vehicle during ascending or descending flight. Although a particular problem of this type is solved readily on a computer, considerable time and many different computer solutions generally are required to determine fully the effects that various parameters have on the problem. An analytical solution to the equations of motion, even an approximate solution, would be a great help in reducing the amount of computer time necessary for such a problem and also would be an aid in extracting aerodynamic force and moment coefficients from the experimental flight characteristics of an ascending or descending vehicle.

In Ref. 1, Friedrich and Dore obtain a Bessel's function solution to the linearized equations of motion for a nonspinning re-entry vehicle and develop an approximate solution for the envelope of the oscillatory motion. A Bessel's function solution for the motion of a nonspinning re-entry vehicle is developed also by Allen.[2] Tobak and Allen[3] extend the Bessel's function solution to the motion of a nonspinning, lifting vehicle which is performing a skip-re-entry maneuver.

Leon[4] obtains a Bessel's function solution for the motion of a spinning re-entry vehicle, but he excludes the effects of aerodynamic damping and Magnus forces and moments. Garber uses a modified *WKB* technique in Ref. 5 to obtain an approximate solution to the linearized equations of motion, including aerodynamic damping, for the spinning re-entry vehicle. White and Steinmetz[6] use the *WKB* method to study the effect of variable aerodynamic coefficients on the linearized motion of a nonspinning vehicle. Murphy[7] uses a quasi-linear technique to study the nonlinear motion of a spinning vehicle, and he includes the Magnus force and moment. The purpose of this paper is to present a unified linear analysis of the motion of an ascending or descending vehicle.

Analysis

This paper will forego a formal development of the equations of motion and will be concerned only with the equations for the lateral oscillatory motion. The justification for separating the oscillatory motion from the translational and quasi-static angular motion is presented in several of the references, e.g., Ref. 1. Excellent developments of the linearized oscillatory equations are presented by Murphy[8] and Nicolaides.[9] The equations of motion developed by Murphy will be used here.

From Ref. 8, the linearized equation for the oscillatory lateral motion referred to a nonrolling (aeroballistic) axis system is

$$\frac{d^2\xi}{dS^2} + \left\{\frac{\rho A}{2m}\left[C_{N\alpha} - 2C_D - k_t^{-2}(C_{M_q} + C_{M\dot{\alpha}})\right] - i\,\frac{p}{V}\,\frac{I_x}{I_y}\right\}\frac{d\xi}{dS} - \left\{\frac{\rho A l C_{M\alpha}}{2I_y} + i\left(\frac{p}{V}\,\frac{I_x}{I_y}\right) \times \left[\frac{\rho A}{2m}\left(C_{N\alpha} - C_D + k_a^{-2}\,C_{M_{p\alpha}}\right)\right]\right\}\xi = 0 \qquad (1)$$

where

$$\xi = \tilde{\beta} + i\tilde{\alpha} = \text{complex angle of yaw}$$

and S is distance travelled from some reference point. The assumptions that have been incorporated in Eq. (1) are: 1) small amplitude motion, $\sin\theta \approx \tan\theta \approx \theta$, $\cos\theta \approx 1$; 2) inertial symmetry, $I_y = I_z = I$; 3) negligible gravity; 4) nonrotating earth; 5) constant mass and moments of inertia; 6) negligible products of aerodynamic coefficients; and 7) negligible distance or time derivatives of aerodynamic coefficients.

It is assumed now that the aerodynamic characteristics of the vehicle are linear with the amplitude of the motion and that the Mach number is large. Under these conditions, the aerodynamic coefficients or derivatives can be considered as constants. Considering the coefficients as constants should yield reasonably accurate results with the possible exception of the Magnus moment coefficient, which will be discussed later.

If canted fins are used to induce spin, the roll rate is, in many instances, approximately proportional to the velocity. For the purpose of this analysis, it will be assumed that

$$p/V = \text{const} \qquad (2)$$

These two assumptions allow Eq. (1) to be written as

$$(d^2\xi/dS^2) + (\rho D - i\,G)(d\xi/dS) - \rho(M + i\,G\,T)\xi = 0 \qquad (3)$$

where

$$D = (A/2m)[(C_{N\alpha} - 2C_D - k_t^{-2}(C_{M_q} + C_{M\dot{\alpha}})] = \text{const} \qquad (4a)$$

$$G = (p/V)(I_x/I) = \text{const} \qquad (4b)$$

$$M = (AlC_{M\alpha}/2I) = \text{const} \qquad (4c)$$

$$T = (A/2m)(C_{N\alpha} - C_D + k_a^{-2}C_{M_{p\alpha}}) = \text{const} \qquad (4d)$$

and

$$\rho = \rho(S) \qquad (4e)$$

The variation of density with altitude can be approximated by the relation

$$\rho = \rho_0 e^{-\beta h} \qquad (5)$$

where h is altitude in feet above some reference point and ρ_0 and β are constant.

It is assumed now that the average flight path can be approximated by a straight line or, at least, straight line segments so that

$$h = S\,\sin\gamma \qquad (6)$$

where γ = const, is the average flight-path angle with respect to the local horizontal. Combining Eqs. (5) and (6) gives

$$\rho = \rho_0 e^{-kS} \qquad (7)$$

where

$$\beta\,\sin\gamma = k = \text{const} \qquad (8)$$

Solution by Assuming an Oscillatory Form

For a first attempt at solving Eq. (3), an oscillatory solution is assumed in the form

$$\xi = E(S)e^{i\phi(S)} \qquad (9)$$

where $E(S)$ is the amplitude or envelope of the motion and $\phi(S)$ is the phase angle or argument. Differentiating Eq. (9) and substituting into Eq. (3) yields

$$\frac{d^2E}{dS^2} - E\left(\frac{d\phi}{dS}\right)^2 + 2\,i\,\frac{dE}{dS}\,\frac{d\phi}{dS} + i\,E\,\frac{d^2\phi}{dS^2} + (\rho D - iG)\left(\frac{dE}{dS} + i\,E\,\frac{d\phi}{dS}\right) - \rho(M + iGT)E = 0 \qquad (10)$$

Separating Eq. (10) into real and imaginary parts gives

$$\frac{d^2E}{dS^2} - E\left(\frac{d\phi}{dS}\right)^2 + \rho D\,\frac{dE}{dS} + GE\,\frac{d\phi}{dS} - \rho ME = 0 \qquad (11)$$

and

$$2\,\frac{dE}{dS}\,\frac{d\phi}{dS} + E\,\frac{d^2\phi}{dS^2} - G\,\frac{dE}{dS} + \rho DE\,\frac{d\phi}{dS} - \rho GTE = 0 \qquad (12)$$

Although Eqs. (11) and (12) are nonlinear and extremely difficult to solve in general, an exact result can be obtained from Eq. (12) for a nonrolling body.

For zero spin, $G = 0$, Eq. (12) reduces to

$$2(dE/dS)(d\phi/dS) + E(d^2\phi/dS^2) + \rho DE(d\phi/dS) = 0 \qquad (13)$$

which can be written as

$$\frac{d}{dS}\left(E^2\,\frac{d\phi}{dS}\right) + \rho D\left(E^2\,\frac{d\phi}{dS}\right) = 0 \qquad (14)$$

Equation (14) can be integrated to give

$$E^2(d\phi/dS) = Ke^{\rho D/k} \qquad (15)$$

Equation (14) is exact within the assumptions of the basic equations of motion and is an interesting relation between the amplitude and frequency of the motion. As a nonrolling body ascends, the moments acting on the body approach zero because of the decreasing air density (neglecting those moments due to gravity gradients, etc.) and, physically, the frequency of the motion also must approach zero. In the limit, as a nonrolling body progresses to a total vacuum, Eq. (15) predicts that the body will tumble, which agrees with physical observations. Without further assumptions, Eq. (15) does not yield any quantitative information about the motion; but it does give an exact criterion for the form of any approximate solution to the equations of motion for a nonrolling body.

Confluent Hypergeometric Solution

Returning to Eq. (3) for the general problem of a spinning body, a new independent variable x is defined as

$$x = \rho D/k = (\rho_0 D/k)e^{-kS} \qquad (16)$$

Equation (3) now reduces to the form

$$x\,\frac{d^2\xi}{dx^2} + \left(1 + \frac{iG}{k} - x\right)\frac{d\xi}{dx} - \left(\frac{M + iGT}{kD}\right)\xi = 0 \qquad (17)$$

Equation (17) is a confluent hypergeometric equation and for $G \neq 0$, has the solution

$$\xi = B_0\Phi(a, c;\ x) + B_1 x^{1-c}\Phi(a - c + 1, 2 - c;\ x) \qquad (18)$$

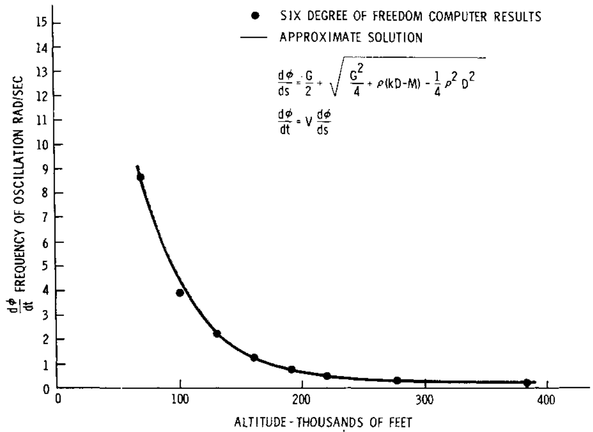

Fig. 1 A comparison of the approximate frequency equation with six-degree-of-freedom computer computations for a typical sounding rocket.

where B_0 and B_1 are arbitrary constants and

$$a = (M + iGT)/kD \tag{19}$$

and

$$c = 1 + i(G/k) \tag{20}$$

The function $\Phi(m, n;\ y)$ is the confluent hypergeometric series

$$\Phi(m, n;\ y) = 1 + \frac{m}{n}\frac{y}{1!} + \frac{m(m+1)}{n(n+1)}\frac{y^2}{2!} + \ldots \tag{21}$$

An examination of Eqs. (19–21) shows that the exact solution, represented by Eq. (18), is an infinite series with complex coefficients. For that reason, the exact solution in terms of the confluent hypergeometric functions is not tractable.

If the vehicle is not rolling, $G = 0$, the coefficients of the differential equation are real, and a stability criterion can be determined for all oscillatory solutions to Eq. (17). However, for a nonrolling body ($G = 0$, $c = 1$), the two solutions in Eq. (18) are not independent and another solution must be determined. Because the independent solutions for the nonrolling body already have been developed by Friedrich and Dore[1] and Allen,[2] only the stability criterion will be considered here.

From Ref. 10, if a, c, and x are real, the envelope of any oscillatory solution to Eq. (17) increases with increasing x if $a > 0$ and $x < c - \frac{1}{2}$, or if $a < 0$ and $x > c - \frac{1}{2}$; the envelope decreases with increasing x if $a > 0$ and $x > c - \frac{1}{2}$, or if $a < 0$ and $x < c - \frac{1}{2}$. It should be noted that increasing altitude is increasing x if the vehicle is dynamically unstable, $D < 0$, and increasing altitude is decreasing x if the body is dynamically stable, $D > 0$.

The critical point in the oscillatory motion of any nonrolling, ascending vehicle is given by $x_{cr} = \rho_{cr}D/k = \frac{1}{2}$, or

$$\rho_{cr} = k/2D = \beta \sin\gamma/2D \tag{22}$$

Table 1 Summary of stability criteria for a nonrolling body exiting the atmosphere[a]

Static stability parameter	Dynamic stability parameter	Amplitude character
$M < 0$	$D > 0$	Convergent for $\rho > \rho_{cr}$ Divergent for $\rho < \rho_{cr}$
$M > 0$	$D > 0$	Convergent for $\rho < \rho_{cr}$ Divergent for $\rho > \rho_{cr}$
$M < 0$	$D < 0$	Divergent for all altitudes
$M > 0$	$D < 0$	Convergent for all altitudes

[a] Applicable only for oscillatory motion.

and the critical altitude can be determined from a density-altitude relation.

The previous stability criterion is summarized in Table I for the various combinations of vehicle parameters. It must be emphasized that this stability criterion is applicable only to oscillatory solutions to Eq. (17) and that particular combinations of vehicle parameters may exclude any possibility of oscillatory motion.[11]

Approximate Solution

Equation (3) can be reduced to its normal form by the transformation

$$\xi = U \exp\left[-\tfrac{1}{2}\int(\rho D - iG)dS\right] = U \exp\tfrac{1}{2}\left[\rho D/k + iGS\right] \tag{23}$$

Substitution of Eq. (23) into Eq. (3) gives

$$(d^2U/dS^2) + U\left\{(G^2/4) + \rho\left[kD - M - iGT + i\,(GD/2)\right] - \tfrac{1}{4}\rho^2D^2\right\} = 0 \tag{24}$$

Application of the *WKB* method[12] to Eq. (24) yields the approximate solution

$$U \approx u^{-1/2}\exp(-i\textstyle\int udS) \tag{25}$$

where

$$u^2 = \left\{(G^2/4) + \rho\left[kD - M - iGT + i\,(GD/2)\right] - \tfrac{1}{4}\rho^2D^2\right\} \tag{26}$$

and the arbitrary constants of the solution have been set equal to 1. Equation (25) is actually the solution to the differential equation

$$(d^2U/dS^2) + u^2(1 - \delta)U = 0 \tag{27}$$

where

$$\delta = 3(du/dS)^2/(4u^4) - (d^2u/dS^2)/(2u^3) \tag{28}$$

Equation (28) represents the error associated with the *WKB* method. The method should give reasonable engineering accuracy when $/\delta/ \lesssim 0.1$. Substitution of Eq. (25) into Eq. (23) yields

$$\xi \approx \frac{\exp\{1/2[(\rho D/k) + iGS] - i\int udS\}}{[u]^{1/2}} \tag{29}$$

It should be noted that Eqs. (26) and (29) represent two independent solutions to the equations of motion. For a nonrolling body, Eq. (29) reduces to

$$\xi = \frac{\exp\{(\rho D/2k) \pm i\int[\rho(kD - M) - \frac{1}{4}\rho^2D^2]^{1/2}dS\}}{[\rho(kD - M) - \frac{1}{4}\rho^2D^2]^{1/4}} \tag{30}$$

If the vehicle characteristics are such that

$$\rho(kD - M) - \tfrac{1}{4}\rho^2D^2 > 0 \tag{31}$$

then Eq. (30) represents two oscillatory solutions of the form assumed in Eq. (9) with

$$E(S) = \frac{\exp(\rho D/2k)}{[\rho(kD - M) - \frac{1}{4}\rho^2D^2]^{1/4}} \tag{32}$$

and

$$\phi(S) = \pm\textstyle\int[\rho(kD - M) - \frac{1}{4}\rho^2D^2]^{1/2}dS \tag{33}$$

Formation of the function $E^2(d\phi/dS)$ from Eqs. (32) and (33) yields

$$E^2\frac{d\phi}{dS} = \pm\left(\exp\frac{\rho D}{k}\right) \times \frac{[\rho(kD - M) - \frac{1}{4}\rho^2D^2]^{1/2}}{[\rho(kD - M) - \frac{1}{4}\rho^2D^2]^{1/2}} = \pm\exp\frac{\rho D}{k} \tag{34}$$

A comparison of Eqs. (15) and (34) shows that the approximate solution is completely compatible with the exact result of Eq. (15).

Differentiation of Eq. (32) yields

$$\frac{dE}{dS} = \frac{\rho}{2}\left(\exp\frac{\rho D}{2k}\right) \times \left\{\frac{M[\rho D - (k/2)] + (k^2D/2) - \frac{5}{4}k\rho D^2 + \frac{1}{4}\rho^2 D^3}{[\rho(kD - M) - \frac{1}{4}\rho^2D^2]^{5/4}}\right\} \quad (35)$$

Setting Eq. (35) equal to zero and avoiding the singular point, $\rho = 0$, yields the critical value of the density from the equation

$$M[\rho_{cr}D - (k/2)] + (k^2D/2) - \tfrac{5}{4}k\rho_{cr}D^2 + \tfrac{1}{4}\rho_{cr}{}^2D^3 = 0 \quad (36)$$

Because $|k| \leq 0.454 \times 10^{-4}$ and $\rho \lesssim 1 \times 10^{-4}$, the last three terms in Eq. (36) are generally negligible and Eq. (36) approximates the exact result of Eq. (22).

Returning to the approximate solution for a rolling body, Eqs. (26) and (29) can be combined to give

$$\xi \approx \frac{\exp\{\frac{1}{2}[(\rho D/k) + iGS] \pm i\int[(G^2/4) + \rho(kD - M - iGT + iGD/2) - \frac{1}{4}\rho^2D^2]^{1/2}\,dS\}}{\{(G^2/4) + \rho[kD - M - iGT + (iGD/2)] - \frac{1}{4}\rho^2D^2\}^{1/4}} \quad (37)$$

If the vehicle characteristics are such that the complex spin term, $G[T - (D/2)]$, is negligible and $(G^2/4) + \rho(kD - M) - \frac{1}{4}\rho^2D^2 > 0$, then Eq. (37) reduces directly to the form $\xi = E(S)\exp i\phi(S)$, where

$$E(S) \approx \frac{\exp(\rho D/2k)}{[(G^2/4) + \rho(kD - M) - \frac{1}{4}\rho^2D^2]^{1/4}} \quad (38)$$

and

$$\phi(S) \approx (GS/2) \pm \int[(G^2/4) + \rho(kD - M) - \tfrac{1}{4}\rho^2D^2]^{1/2}dS \quad (39)$$

It should be noted that Eq. (38) represents the amplitude of each component of the total solution and, in general, does not represent the total amplitude of the motion. Differentiating Eq. (38) gives

$$\frac{dE}{dS} = \frac{\rho}{2}\left(\exp\frac{\rho D}{2k}\right) \times \left\{\frac{-D[(G^2/4) - \rho M] - (kM/2) - \frac{5}{4}k\rho D^2 + \frac{1}{2}k^2D + \frac{1}{4}\rho^2D^3}{[(G^2/4) + \rho(kD - M) - \frac{1}{4}\rho^2D^2]^{5/4}}\right\} \quad (40)$$

Setting Eq. (40) equal to zero gives the nontrivial critical point from the equation

$$-D[(G^2/4) - \rho_{cr}M] - (kM/2) - \tfrac{5}{4}k\rho_{cr}D^2 + \tfrac{1}{2}k^2D + \tfrac{1}{4}\rho_{cr}{}^2D^3 = 0 \quad (41)$$

In general, the critical value of the density from Eq. (41) will not be significantly different from the critical density from Eq. (22) for a nonrolling vehicle. Diffferentiating Eq. (39) gives

$$(d\phi/dS) = (G/2) \pm [(G^2/4) + \rho(kD - M) - \tfrac{1}{4}\rho^2D^2]^{1/2} \quad (42)$$

which is the frequency of the particular mode of motion, depending upon the positive or negative sign. Equation (38) can be nondimensionalized by forming the amplitude ratio

$$\frac{E}{E_{\text{vacuum}}} = \frac{\exp(\rho D/2k)}{\{1 + [4\,\rho(kD - M)/G^2] - (\rho^2D^2/G^2)\}^{1/4}} \quad (43)$$

Equations (42) and (43) are compared with six-degrees-of-freedom computer computations in Figs. 1 and 2, respectively, for the typical sounding rocket vehicle described in Fig. 3. The accuracy of the approximate solution is good.

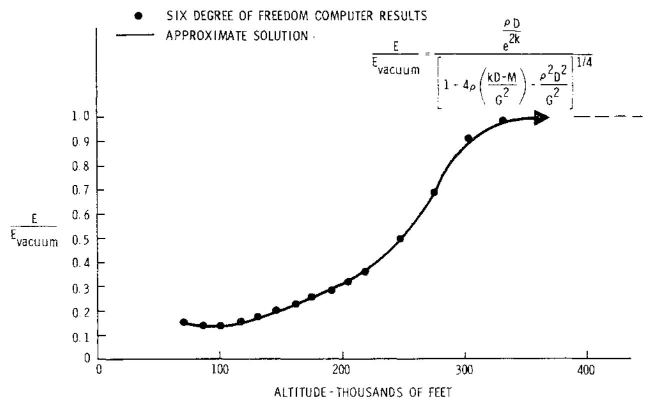

Fig. 2 A comparison of the approximate amplitude solution with six-degree-of-freedom computer computations for a typical sounding rocket.

Magnus Moment Coefficient

It was stated at the beginning of this analysis that the aerodynamic coefficients would be assumed constant, with the possible exception of the Magnus moment coefficient. The implicit exact solution for a nonrolling body, Eq. (15), requires only that the damping parameter D be a constant. The exact solution for a rolling body, Eq. (18), requires that all of the aerodynamic coefficients be constant. The approximate solution is valid for variable aerodynamic coefficients so long as the coefficients are slowly varying (derivatives of coefficients are negligible) and independent of the angle of attack (linear aerodynamic characteristics).

Since this analysis is concerned with high-speed flight, it seems reasonable to assume that the coefficients are independent of Mach number. In spite of this assumption, the Magnus moment still poses a somewhat interesting dilemma for the problem of ascending or descending flight. For a vehicle entering or leaving the atmosphere at high speeds, the unit Reynolds number varies over a range from zero to several million per foot. With this wide variation of Reynolds number, the use of a constant Magnus moment coefficient in a stability analysis of a spinning vehicle in ascending or descending flight is not very realistic.

Unfortunately, the admission of a variable Magnus moment coefficient, dependent upon Reynolds number, complicates

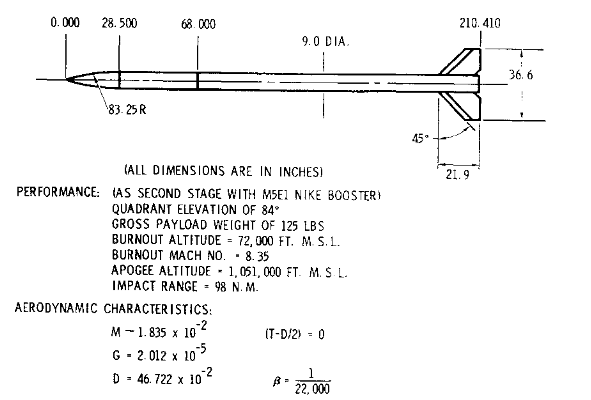

Fig. 3 Typical sounding rocket configuration (Nitehawk 9-in. second stage).

the problem because the quantitative nature of this variation is unknown generally. Platou[13] illustrates several of the causes and complexities of the Magnus moment and gives some methods of estimating its magnitude.

The approximate solution for the general motion is, from Eqs. (26) and (29),

$$\xi \approx \frac{\exp\{\frac{1}{2}[(\rho D/k) + iGS] - i\int u ds\}}{[u]^{1/2}} \quad (44)$$

where

$$u^2 = \{(G^2/4) + \rho(kD - M - iGT + iGD/2) - \tfrac{1}{4}\rho^2 D^2\} \quad (45)$$

Let

$$u = v + iw \quad (46)$$

Substituting Eq. (46) into Eq. (45) and separating the real and imaginary parts gives

$$v^2 - w^2 = (G^2/4) - \rho M + k\rho D - \tfrac{1}{4}\rho^2 D^2 \quad (47)$$

and

$$2vw = -G\rho[T - (D/2)] \quad (48)$$

Solving Eq. (48) for v and substituting into Eq. (47) gives

$$(G^2\rho^2/4w^2)[T - (D/2)]^2 - w^2 = (G^2/4) - \rho M + k\rho D - \tfrac{1}{4}\rho^2 D^2 \quad (49)$$

Because $\rho \ll 1$ and $|k| \ll 1$, the last two terms in Eq. (49) are generally negligible. Under this condition, Eq. (49) reduces to

$$(G^2\rho^2/4w^2)[T - (D/2)]^2 - w^2 \approx (G^2/4) - \rho M \quad (50)$$

The gyroscopic term $G^2/4$ is retained in the right side of Eq. (50) because it has a very significant effect on the motion of a body exiting the atmosphere.

Equation (50) is a quadratic equation in w^2 and can be solved to give

$$w^2 \approx \frac{1}{2}\left(\frac{G^2}{4} - \rho M\right)\left(-1 \pm \left\{1 + \frac{\rho^2 G^2[T - (D/2)]^2}{[(G^2/4) - \rho M]^2}\right\}^{1/2}\right) \quad (51)$$

For a statically stable vehicle, $M < 0$, the positive sign in Eq. (51) is selected because w^2 must be positive. Thus,

$$w^2 \approx \frac{1}{2}\left(\frac{G^2}{4} - \rho M\right)\left(-1 + \left\{1 + \frac{\rho^2 G^2[T - (D/2)]^2}{[(G^2/4) - \rho M]^2}\right\}^{1/2}\right) \quad (52)$$

If

$$z = \frac{\rho^2 G^2[T - (D/2)]^2}{[(G^2/4) - \rho M]^2} < 1$$

Eq. (52) can be expanded using the binomial theorem to give

$$w^2 \approx \tfrac{1}{2}[(G^2/4) - \rho M]\{-1 + 1 + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \tfrac{1}{16}z^3 + \ldots\} \quad (53)$$

Equation (53) can be written as

$$w^2 \approx \frac{\rho^2 G^2[T - D/2]^2}{4[(G^2/4) - \rho M]}\left\{1 - \frac{z}{4} + \frac{z^2}{8} + \ldots\right\} \quad (54)$$

If $z \ll 1$, Eq. (54) reduces to

$$w^2 \approx \frac{\rho^2 G^2[T - (D/2)]^2}{4[(G^2/4) - \rho M]} \quad (55)$$

or

$$w \approx \pm \frac{\rho G[T - (D/2)]}{2[(G^2/4) - \rho M]^{1/2}} \quad (56)$$

Substitution of Eq. (56) into Eq. (48) yields

$$v \approx \mp[(G^2/4) - \rho M]^{1/2} \quad (57)$$

The approximate solution for the general motion can be written as

$$\xi \approx \frac{e\{\frac{1}{2}[(\rho D/k) + iGS] - i\int(v + iw)dS\}}{[v + iw]^{1/2}} \quad (58)$$

or

$$\xi \approx \frac{\exp(-i\theta/2)\exp\{[(\rho D/2k) - \int w dS] + i[(GS/2) - \int v dS]\}}{[v^2 + w^2]^{1/4}} \quad (59)$$

where

$$\theta = \tan^{-1}(w/v) \quad (60)$$

$$v \approx \mp[(G^2/4) - \rho M]^{1/2} \quad (61)$$

and

$$w \approx \pm \frac{\rho G[T - (D/2)]}{2[(G^2/4) - \rho M]^{1/2}} \quad (62)$$

From Eqs. (59) and (62) the predominant effect of the Magnus moment is in the damping of the motion as it is in the constant density problem. The effect of the Magnus moment on the frequency has been suppressed through the approximation from Eq. (54) to Eq. (55), i.e.,

$$\frac{\rho^2 G^2[T - (D/2)]^2}{[(G^2/4) - \rho M]^2} \ll 1 \quad (63)$$

This approximation can be written as

$$\frac{\rho^2 G^2[T - (D/2)]^2}{[(G^2/4) - \rho M]^2} = \frac{4w^2}{v^2} \ll 1 \quad (64)$$

which can be taken as

$$|w| \ll |v| \quad (65)$$

Equation (64) says, essentially, that the ratio of the Magnus moment to the static moment must be small for the approximation to be valid. An interesting contribution of the Magnus moment in the variable density problem is a phase shift of the motion through the term $\exp -i\theta/2$ in Eq. (59). A phase shift of the motion has been observed in experimental attitude data from tests of a two-stage sounding rocket, but it has not been identified positively as a Magnus moment effect. To emphasize the two independent solutions represented by Eqs. (59, 61, and 62), the approximate solution is written finally as†

$$\xi \approx \{\exp[(\rho D/2^k) - i\theta/2]\}/[v^2 + w^2]^{1/4} \times (K_0 \exp\{\int w ds + i[(GS/2) + \int v dS]\} + K_1 \exp\{-\int w dS + i[(GS/2) - \int v dS]\}) \quad (66)$$

where

$$v \approx +[(G^2/4) - \rho M]^{1/2} \quad (67)$$

† The positive value has been selected for v and w, and the equation has been expanded to allow for this selection.

$$w \approx + \frac{\rho G[T - (D/2)]}{2[(G^2/4) - \rho M]^{1/2}} \tag{68}$$

and

$$\theta = \tan^{-1}(-w/v) \tag{69}$$

It is interesting to note several characteristics of Eq. (66). First, as a statically stable rolling body ascends, the frequencies of both modes of motion decrease, with the nutational (K_0) frequency approaching its vacuum value of G and the precessional (K_1) frequency approaching its vacuum value of zero. If the Magnus moment term w is negligible, then both modes of motion will grow or damp at the same rate. In general, the complex plane plot of Eq. (66) for a typical body will start as an ellipse, precessing about $\xi = 0$ and will degenerate to a circle with its center located off of the ξ origin. The center of this degenerate circular motion is the precessional arm that has stopped rotating.

When the Magnus moment terms are retained in Eq. (66), the two modes of motion no longer grow or damp at the same rate. The Magnus moment term tends to stabilize one mode of motion and destabilize the other mode while not affecting the frequencies of the modes. If the Magnus moment term is not large enough to change the damping trends of the motion, the complex plane representation of Eq. (66) would be similar to that mentioned previously. If, however, the Magnus moment term is capable of completely damping the precessional mode of motion, the complex plane plot will degenerate to a circle centered at the ξ origin. If the Magnus moment term is capable of completely damping the nutational mode of motion, the complex plane plot degenerates to a point located off of the ξ origin, i.e., the tip of the precessional vector which has stopped rotating.

Conclusions

An analysis of the linearized differential equation for the complex yaw angle has yielded the following solutions for the motion of a coasting vehicle ascending through an exponential atmosphere at high speeds:

1) Nonrolling body, exact solution: If $\xi = E(S) \exp i\phi(S)$, then $E^2(d\phi/dS) = K \exp(\rho D/k)$; 2) rolling body, exact solution:

$$\xi = B_0\Phi(a, c;\ x) + B_1 x^{1-c}\Phi(a - c + 1, 2 - c;\ x)$$

where the Φ's are confluent, hypergeometric functions, and a and c are complex numbers; 3) rolling body, approximate solution:

$$\xi \approx \frac{\exp[(\rho D/2k) - i\theta/2]}{[v^2 + w^2]^{1/4}} \times \left(K_0 \exp\left[\int w dS + i\left(\frac{GS}{2} + \int v dS\right)\right] + K_1 \exp\left[-\int w dS + i\left(\frac{GS}{2} - \int v dS\right)\right]\right)$$

The exact solutions are restricted to the assumption of constant aerodynamic coefficients, whereas the approximate solution is applicable for variable aerodynamic coefficients. Although the ascending mode of the trajectory has been emphasized in this report, the solutions can be extended readily to the descending mode whenever the angular misalignment of the vehicle from its trajectory is small.

From these solutions for the motion of an ascending vehicle, there exists a critical value of density or altitude at which the damping characteristics of the motion change sign, regardless of the stability characteristics of the vehicle. In some instances, this critical value of the density may be trivial or physically impossible, e.g., $\rho < 0$, etc.

The approximate solution for a rolling body with zero Magnus moment contribution has been compared with six-degrees-of-freedom computer computations. The agreement was good. It is felt that the approximate solution will give good engineering accuracy in a great many applications and that it represents an excellent analytical tool for studying the effects of various aerodynamic forces and moments on the motion.

References

[1] Friedrich, H. R. and Dore, F. J., "The dynamic motion of a missile descending through the atmosphere," J. Aeronaut. Sci. **22,** 628–632 (1955).

[2] Allen, H. J., "Motion of a ballistic missile angularly misaligned with the flight path upon entering the atmosphere and its effect upon aerodynamic heating, aerodynamic loads, and miss distance," NACA TN 4048 (October 1957).

[3] Tobak, M. and Allen, H. J., "Dynamic stability of vehicles traversing ascending or descending paths through the atmosphere," NACA TN 4275 (July 1958).

[4] Leon, H. I., "Angle of attack convergence of a spinning missile descending through the atmosphere," J. Aeronaut. Sci. **25,** 480–489 (1958).

[5] Garber, T. B., "On the rotation of motion of a body re-entering the atmosphere," J. Aeronaut. Sci. **26,** 443–449 (1959).

[6] White, F. M., Jr. and Steinmetz, W. J., "Dynamic stability of a missile under variable velocity conditions," Sandia Corp., SCTM 122-63(71) (June 1963).

[7] Murphy, C. H., "Effect of varying air density on the nonlinear pitching and yawing motion of a symmetric missile," Ballistic Research Lab. Rept. 1162 (February 1962).

[8] Murphy, C. H., "Free flight motion of symmetric missiles," Ballistic Research Lab. Rept. 1216 (July 1963).

[9] Nicolaides, J. D., "Missile flight and astrodynamics," Bureau of Weapons, Department of the Navy, TN 100-A (1959–1961).

[10] Bateman, H., *Higher Transcendental Functions* (McGraw-Hill Book Co. Inc., New York, 1953), Vol. 1, Chap. VI, p. 291.

[11] Bellman, R., *Stability Theory of Differential Equations* (McGraw-Hill Book Co. Inc., New York, 1953), Chap. VI, pp. 119–122.

[12] White, F. M., Jr., "The use of the WKB method in applied mechanics," *Proceedings of the First Southwest Symposium on Applied Mechanics* (Gatlinburg, Tenn., May 1962).

[13] Platou, S., "Magnus characterization of finned and nonfinned projectiles," AIAA J. **3,** 83–90 (1965);